where $(\lambda - c)(\mu - c) = h^2$, $\lambda\mu = l^2$, $\lambda \geqq \mu$. If l is assumed $\geqq c + h$, as is natural, λ and μ are real, unequal, and greater than c. The result of the transformation is

$$A = 2(\lambda - \mu) \int_{\sqrt{\frac{\mu - c}{\lambda - c}}}^{\sqrt{\frac{\mu}{\lambda}}} \frac{\sqrt{(\lambda - c)v^2 - (\mu - c)}}{\sqrt{\mu - \lambda v^2}} \cdot \frac{\lambda v + \mu}{(v + 1)^3} dv.$$

The final result will be simplified if we transform again, putting

$$v = \frac{\sqrt{\mu(\mu - c)}}{\sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}},$$

which gives

$$A = 2c(\mu - c) \int_0^1 \frac{(3\lambda c + 3\mu c - 2\lambda\mu - 4c^2)\mu^2 + (\lambda\mu + 5\mu c - 3\mu^2 - 3\lambda c)\mu c w^2 + (\lambda - \mu)\mu c^2 w^4}{(\mu - cw^2)^3 \sqrt{1 - w^2} \sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}} \cdot w^2 dw$$

$$+ 2c(\mu - c)\sqrt{\mu(\mu - c)} \int_0^1 \frac{(\lambda + \mu - 4c)\mu - (\lambda - 3\mu)cw^2}{(\mu - cw^2)^3 \sqrt{1 - w^2}} \cdot w^2 dw.$$

These integrals are respectively elliptic and circular. On making the necessary reductions, we find, after much calculation,

$$A = c(\mu - c)\sqrt{\frac{\mu}{\lambda - c}} F\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) - c\sqrt{\mu(\lambda - c)} E\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right)$$

$$- \frac{c(\mu - c)(\lambda + \mu - c)}{\sqrt{\mu(\lambda - c)}} \Pi\left(-\frac{c}{\mu}, \sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) + \frac{\pi}{2} \cdot c(\lambda + \mu - c).$$

2697 [April, 1918]. Proposed by H. S. UHLER, Yale University.

Show how to reduce the left-hand members of the following identities to their respective right members:

$$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \sin^2 y, \tag{1}$$

$$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = \sin \tfrac{1}{2}y \sin y, \tag{2}$$

$$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = \sin \tfrac{1}{2}y \sin y. \tag{3}$$

SOLUTION BY POLYCARP HANSEN, St. John's University, Collegeville, Minn.

The terms of the left-hand members can be expressed as follows:

(1)
$$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \frac{1 - \cos (2x + y)}{2} + \frac{1}{2}[\cos (2x + y) - \cos 2y] = \frac{1 - \cos 2y}{2} = \sin^2 y.$$

(2)
$$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = -\tfrac{1}{2}[\cos (2x + \tfrac{3}{2}y) - \cos \tfrac{1}{2}y] + \tfrac{1}{2}[\cos (2x + \tfrac{3}{2}y) - \cos (-\tfrac{3}{2}y)] = \tfrac{1}{2}[\cos \tfrac{1}{2}y - \cos (-\tfrac{3}{2}y)] = \sin \tfrac{1}{2}y \sin y.$$

(3)
$$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = -\tfrac{1}{2}[\cos (2x + \tfrac{1}{2}y) - \cos (-\tfrac{1}{2}y)] + \tfrac{1}{2}[\cos (2x + \tfrac{1}{2}y) - \cos (-\tfrac{3}{2}y)] = \sin \tfrac{1}{2}y \sin y.$$

Also solved by R. B. WILDERMUTH, JEROME J. JULIAN, KATHERINE S. ARNOLD, R. M. MATHEWS, H. L. OLSON, H. E. CARLSON, A. T. DINEEN, R. C. COLWELL, and ROGER A. JOHNSON.

2698 [April, 1918]. Proposed by WARREN WEAVER, Throop College of Technology, Pasadena, California.

An urn contains N balls, numbered from 1 to N. Of these n are drawn out and are arranged linearly according to the numbers on each. A certain ball is observed to be the kth in this line. What is the most probable number written on this ball?

I. SOLUTION BY HARRY M. ROESER, BUREAU OF STANDARDS, Washington, D. C.

The total number of selections of n balls from N balls is

$$U = \frac{\underline{|N}}{\underline{|n}\,\underline{|N-n}}.$$

The least number that can be in the kth place is k. In the urn there are $(N - k)$ numbers greater than k and $(k - 1)$ numbers less than k. The numbers less than k can be arranged in groups of $(k - 1)$ each in 1 way. After this is done the numbers greater than k can be arranged in groups of $(n - k)$ numbers each in $\dfrac{\underline{|N-k}}{\underline{|n-k}\,\underline{|N-n}}$ ways. Therefore the total number of sequences with k in the kth place is

$$u_0 = \frac{1 \cdot \underline{|N-k}}{\underline{|n-k}\,\underline{|N-n}}$$

and by definition the probability that k will be in the kth place is u_0/U.

The next lowest number that can be in the kth place is $(k + 1)$. The k numbers less than $(k + 1)$ can be arranged in groups of $(k - 1)$ numbers each in

$$\frac{\underline{|k}}{\underline{|k-1}} = k$$

ways. After this is done the $N - k - 1$ numbers greater than $(k + 1)$ can be arranged in groups of $(n - k)$ numbers each in $\dfrac{\underline{|N-k-1}}{\underline{|n-k}\,\underline{|N-n-1}}$ ways. Therefore, the total number of sequences with $(k + 1)$ in the kth place is

$$u_1 = \frac{\underline{|N-k-1}}{\underline{|n-k}\,\underline{|N-n-1}} \cdot k$$

and the probability that $(k + 1)$ will be in the kth place is u_1/U.

By similar reasoning it readily follows that the respective probabilities that k, $(k + 1)$, $(k + 2)$, $\cdots$ $(k + i)$, $\cdots$, are in the kth place are given by the successive terms of the series,

$$\frac{1}{U}[u_0 + u_1 + u_2 + \cdots u_i + \cdots] = \frac{\underline{|N-n}\,\underline{|n}}{\underline{|N}}\Bigg[\frac{\underline{|N-k}}{\underline{|n-k}\,\underline{|N-n}} + \frac{\underline{|N-k-1}}{\underline{|n-k}\,\underline{|N-n-1}} \cdot k$$
$$+ \frac{\underline{|N-k-2}}{\underline{|n-k}\,\underline{|N-n-2}} \cdot \frac{(k+1)\cdot k}{\underline{|2}} + \cdots \frac{\underline{|N-k-i}}{\underline{|n-k}\,\underline{|N-n-i}} \cdot \frac{(k+i-1)\cdots k}{\underline{|i}} + \cdots\Bigg]$$

and the number most probably in the kth place is given by adding to k the value of i corresponding to the largest term of this series.

In the above series the ratio of the term corresponding to $(i + 1)$ to the term corresponding to i is

$$m = \frac{N-n-i}{N-k-i} \cdot \frac{k+i}{i+1}.$$

Setting $m \gtreqqless 1$ we find by reduction of the inequalities that

$$i \lesseqqgtr N\frac{(k-1)}{n-1} - k$$

and, therefore, the number most likely to be found in the kth place is given by the integral part of $(k + i + 1)$ or the integral part of $N\dfrac{k-1}{n-1} + 1$.

If $N\dfrac{k-1}{n-1}$ is an integer there will be two successive numbers $N\dfrac{k-1}{n-1}$ and $N\dfrac{k-1}{n-1} + 1$ each equally likely to be found in the kth place and more likely to be found there than any other numbers in the urn.

II. Mr. C. F. GUMMER solved the problem in a similar manner and added: It may be of interest to find the expectation. This will be

$$\sum_{r=k}^{r=N-n+k} \frac{\binom{r-1}{k-1}\binom{N-r}{n-k}}{\binom{N}{n}} r = \frac{k\,\underline{|n}}{\underline{|N}} f(N, n, k),$$

where r is the number on the kth ball, and

$$f(N, n, k) = \sum_{r=k}^{r=N-n+k} \binom{N-n}{r-k} \frac{\underline{|r}}{\underline{|k}} \frac{\underline{|N-r}}{\underline{|n-k}}.$$

From the relation

$$\binom{N-n}{r-k} = \binom{N-n-1}{r-k-1} + \binom{N-n-1}{r-k},$$

it follows that

$$f(N, n, k) = (n-k+1)f(N, n+1, k) + (k+1)f(N, n+1, k+1). \tag{1}$$

Now $f(N, N, k) = 1$, being independent of k. Therefore, by (1), $f(N, N-1, k) = N+1$, also independent of k, and finally $f(N, n, k) = (N+1)N\cdots(n+2)$. Hence, the expectation for the kth ball is $k(N+1)/(n+1)$.

2699 [May, 1918]. Proposed by the late ROGER E. MOORE, University of Wisconsin.

Show that if $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r, and c_k denotes the kth binomial coefficient in the expansion of $(a-b)^n$, n being a positive integer,

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad \text{if} \qquad n > r.$$

SOLUTION BY ELBERT H. CLARKE, Hiram College.

Let d_0 be the first term in the arithmetic progression and let $d_1, \cdots, d_r$ denote the initial difference of each order. Using the usual abbreviated notation for binomial coefficients, we write

$$a_k^{(r)} = \binom{k-1}{0} d_0 + \binom{k-1}{1} d_1 + \binom{k-1}{2} d_2 + \cdots + \binom{k-1}{r} d_r,$$

$$c_k = (-1)^{k-1}\binom{n}{k-1}$$

and

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = \sum_{i=0}^{r} d_i \sum_{k=1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1}.$$

Consider the inner sum. Since $\binom{k-1}{i} = 0$, for $k < i+1$,

$$\sum_{k=1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1} = \sum_{k=i+1}^{n+1} (-1)^{k-1}\binom{k-1}{i}\binom{n}{k-1},$$

and the latter expression easily becomes

$$\binom{n}{i}\sum_{k=i+1}^{n+1} (-1)^{k-1}\binom{n-i}{k-(i+1)}.$$

Now put $k-i-1 = t$ and we have

$$\binom{n}{i}\sum_{t=0}^{n-i} (-1)^{t+i}\binom{n-i}{t}.$$

But the expression under summation is simply $(-1)^i(1-1)^{n-i}$. Hence, the coefficient of every d_i is zero. Therefore,

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad n > r.$$